Analytic Results for MHV Wilson Loops

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Abstract

We obtain concise analytic formulae for Wilson loops computed on special n-point polygonal contours through two-loops in weakly coupled $\mathcal{N} = 4$ supersymmetric gauge theory. The contours we consider can be embedded into a $(1 + 1)$-dimensional subspace of the 4-dimensional gauge theory, corresponding to the boundary of the AdS_3 on the string theory side. Our analytic results hold for any number of edges, thus generalising to arbitrary n the recently derived expressions for 2-dimensional octagons. These polygonal Wilson loops have been conjectured to be equivalent to MHV scattering amplitudes in planar $\mathcal{N} = 4$ SYM.

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1 Introduction

An ambitious goal of solving the $\mathcal{N} = 4$ supersymmetric Yang-Mills theory (SYM) should include a full understanding of the structure of dynamical quantities in this theory, in particular the S-matrix, or scattering amplitudes.

It was shown in [1] that at strong coupling, scattering amplitudes can be determined through a calculation of a Wilson loop

$$W_n := W[\mathcal{C}_n] = \operatorname{Tr} \mathcal{P} \exp\left[ig \oint_{\mathcal{C}_n} d\tau \; \dot{x}^\mu(\tau) A_\mu(x(\tau))\right] \; , \qquad (1.1)$$

with a light-like n-edged polygonal contour $\mathcal{C}_n$ obtained by attaching the momenta of the scattered particles $p_1, \ldots, p_n$ one after the other, following the order of the colour generators in the colour-ordered scattering amplitude. The vertices, x_i, of the polygon are related to the external momenta via $p_i = x_i - x_{i+1}$, where $x_{n+1} = x_1$.

Since then it has been conjectured that at any value of the coupling in planar $\mathcal{N} = 4$ SYM there is a non-trivial relation between scattering amplitudes and Wilson loops [1–3]. The weak coupling relation between MHV scattering amplitudes and Wilson loops has been supported by an increasing amount of evidence [1–8]. In perturbation theory the number of different Feynman topologies in the Wilson loop does not grow with the number of external particles (edges) n [8] and is much smaller than the corresponding number of integrals for the n-point amplitude [9]. This, together with the fact that the integrals themselves are also more straightforward to compute, makes the computation of W_n considerably more attractive practically than that of the amplitudes.

In practice, the duality between MHV amplitudes and Wilson loops beyond one-loop is understood in terms of the remainder function. The remainder function of the amplitude $\mathcal{R}_n$ (or of the Wilson loop $\mathcal{R}_n^{WL}$) is defined as the difference between the logarithm of the amplitude $\mathcal{M}_n$ (or Wilson loop W_n) and the known BDS expression obtained in [10, 11], so that

$$\mathcal{R}_n = \log(\mathcal{M}_n) - (BDS)_n \; , \qquad \mathcal{R}_n^{WL} = \log(W_n) - (BDS)_n^{WL} . \qquad (1.2)$$

Here $\mathcal{M}_n$ is the colour-ordered MHV amplitude normalised by the tree-level result, $\mathcal{M}_n := \mathcal{A}_n^{\mathrm{MHV}}/\mathcal{A}_n^{\mathrm{MHV\,tree}}$ The Wilson loop possesses an anomalous conformal symmetry and the remainder function constitutes the correctly regularised, conformally invariant part of the Wilson loop, reducing the number of independent variables down to the conformally invariant cross-ratios [5]. The Wilson loop/amplitude duality states that the two remainder functions are identical [6–8], $\mathcal{R}_n = \mathcal{R}_n^{WL}$.

Ref. [8] has assembled a general algorithm for computing Wilson loops W_n for arbitrary n at two loops and has studied their multi-collinear limits. The actual

computations in [8] were carried out numerically for up to $n = 8$ edges in general kinematics, and in [12, 13] for up to $n = 30$ edges for special families of regular polygons. Fully analytic two-loop calculations of W_n were pioneered for the case of the hexagon in [14,15] and derived using quasi-multi-Regge kinematics at intermediate stages to simplify the computation. The result at $n = 6$ was expressed in terms of a 17-pages long linear combination of generalised polylogarithm functions of uniform transcendental weight four (or alternatively it was also recast in [16] in terms of one-dimensional integrals.) Following these exciting developments, the authors of [17] were able to find another representation of the hexagon result at two-loops in terms of a remarkably simple compact expression involving only Li_m functions with $m \leq 4$ and logarithms (again of total weight four).

Another recent inspiring achievement stems from the study of Wilson loops whose contours are chosen to lie in a $(1+1)$-dimensional subspace of Minkowski space-time. At strong coupling, this corresponds to Wilson loops embedded into the boundary of AdS_3 target space of the dual string theory. Alday and Maldacena in [18] have computed the corresponding Wilson loop at strong coupling using an auxiliary integrable system, and have expressed their result in terms of a simple one-dimensional integral. In Ref. [12] octagonal Wilson loops in the same kinematics were evaluated numerically also at weak coupling and it was suggested that there are pronounced similarities between the weak and the strong coupling contributions in this special 2-dimensional kinematics. The analytic result for $(1 + 1)$-dimensional octagon at two loops was very recently derived by the authors of [19] following the technology established in their earlier papers [14, 15]. The striking result of [19] is that all the complications related to multiple occurrences of generalised polylogarithms have mutually cancelled in the final expression for the $(1 + 1)$-dimensional octagon, leaving only a product of four logarithms plus constant terms in the two loop expression for the Wilson loop remainder function. The striking simplicity of this analytic end result for the $(1+1)$-dimensional octagon makes it even simpler than the corresponding strong coupling result derived in [18] (even though numerically the two functions remain very close).

The motivation of this paper is to investigate whether wider classes of Wilson loops exist to the eight point case of [19], but which exhibit a similarly simple analytic form at weak coupling. We will show that this is indeed the case. We shall consider Wilson loops with an arbitrary number n of light-like edges and will require that they can be embedded into a $(1+1)$-dimensional subspace of Minkowski space. These Wilson loops are also conformally equivalent to those with contours in $(2 + 1)$-dimensions whose spatial projection circumscribes the unit circle [18]. For these Wilson loops we will find that all the polylogarithms appearing at one- and two-loops disappear from final answers and the entire two loop contribution is described by a weight-four function composed entirely of logarithms (and constant terms).

Our programme is in two parts. First we construct an analytic expression for

$\log W_n$ which is required to satisfy certain precise criteria. Then we verify that this analytic expression does agree with the numerical evaluation of $\log W_n$ which we carry out following the algorithm of [8]. Since our numerical computations can be carried out at a press of a button for any kinematics and any n-polygons and with an ever increasing accuracy, this is where the programme of numerically evaluating Wilson loops (or remainders) starts paying off.

To come up with the analytic ansatz in the first place, we will employ the following strategy. Firstly we assume that only logarithms can appear and none of the complicated polylogarithms are allowed in $\log W_n$ for the $(1+1)$-dimensional contours. Secondly we assume that the arguments of these logarithms for the conformally-invariant part of the answer[1] are the cross-ratios themselves, and the total expression at each loop-level has the appropriate uniform transcendental weight.

We then start assembling various pieces of evidence for the resulting ansatz, based on simplifying the known one-loop expressions at all n of [20] together with employing the recently found eight-point two-loop result of [19]. Further evidence suggesting the appearance of simple cross-ratios only as arguments of the logarithms (and not more complicated functions of cross-ratios) can be found from the recent tremendously simplified form of the six-point remainder function [17]. At first sight this result seems to suggest the opposite, since there one finds highly complicated functions (involving square roots) of the simple cross-ratios appearing as arguments of the polylogarithms. However these complicated functions were interpreted in terms of momentum twistors. In the special kinematics in $(1+1)$-dimensions the momentum twistors go to simple products of space-time coordinates, and so this observation also points to the fact that only cross-ratios themselves should appear as arguments in the specialised kinematics.

This paper is organised as follows. In section 2 we introduce the two-dimensional kinematics and define the cross-ratios relevant to polygonal contours with n light-like edges. In section 3 we show that Wilson loops in this kinematics depend only on simple logarithms at one loop by simplifying the one-loop expression. In section 4 we move on to the discussion of two-loop results at $n = 8$, $n = 10$ and $n = 12$ points. The corresponding analytic expressions are given in Eqs. (4.2),(4.3) and (4.8). We then construct the general analytic expression valid for any n in (4.11). All of these results satisfy stringent tests from the multi-collinear limits and are verified numerically following the methods developed in [8]. Finally we also check consistency of these results with the regular polygons computations performed earlier in [12]. We present our conclusions in section 5 where we also comment on the possible structure of higher loop contributions beyond two loops. An Appendix discusses the consistency of our two loop expression with multi-collinear limits in more detail.

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[1]This means the Li_2 part of the one-loop answer and the remainder function $\mathcal{R}_n$ at two loops.

2 Two-dimensional kinematics

The remainder function is a conformally-invariant object. As such it depends on the kinematics only through conformally-invariant cross-ratios [2, 5]. In this section we will define these cross-ratios in the $(1+1)$-dimensional case. This is relevant for polygonal contours with n light-like edges which can be embedded into the boundary of the three-dimensional anti-de-Sitter subspace, AdS_3 of the dual string theory $AdS_5 \times S^5$ target space.

For a polygonal contour with n light-like edges, in general, there are $n(n-5)/2$ independent conformal cross-ratios (if we do not, as in [8], impose the Gram determinant constraints). This is the same as the number as of two-mass easy boxes. As always, we use the basis for the cross ratios $u_{i,j}$,

$$u_{ij} = \frac{x_{ij+1}^2 x_{i+1j}^2}{x_{ij}^2 x_{i+1j+1}^2} \ , \tag{2.1}$$

which 'connect' edges i and j, as shown in Figure 1. Here x_i are the vertices of the polygonal contour of the Wilson loop, and $j \geq i + 3$ modulo n.
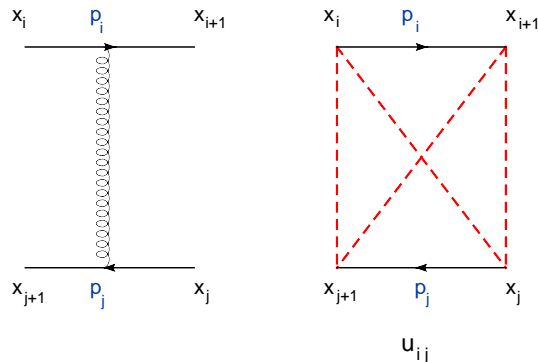

Figure 1: *The left figure shows the one-loop Wilson loop diagram which gives the finite part of the two-mass easy box function with massless momenta p_i, p_j as in [3]. On the right we represent the corresponding cross-ratio u_{ij}, the red dashed lines depicting the factors x_{ij}^2, x_{i+1j+1}^2, x_{i+1j}^2, x_{ij+1}^2 in the definition of u_{ij} in (2.1). Later we will represent the cross-ratio as a single line stretched between edges i and j similarly to the gluon propagator.*

For the Wilson loop contour to be embeddable into two space-time dimensions[2] the number of edges n must be even. In two dimensions the number of independent

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[2]The Wilson loop itself of course is calculated in $D = 4 - 2\epsilon$ dimensions, only its contour is now embedded into $(1+1)$-dimensions which can be thought of as the boundary of AdS_3.

cross-ratios reduces and they have to satisfy the following conditions,

$$
\begin{aligned}
u_{i,i+\text{odd}} &= 1 \\
u_{2i+1,2j+1} &= u_{ij}^{+} \qquad 2 \le (i-j) \bmod n/2 \le n/2 - 2 \\
u_{2i,2j} &= u_{ij}^{-}
\end{aligned}
\tag{2.2}
$$

where we have defined the two dimensional cross-ratios $u_{ij}^{\pm}$ in terms of light-cone coordinates as will be explained momentarily. Clearly, just as the indices of the cross-ratios u_{ij} are defined mod n, the two-dimensional light-cone cross-ratios $u_{ij}^{\pm}$ have indices defined mod $n/2$. The sequence of $(1+1)$-dimensional light-like edges of the contour must have a zig-zag form as in [18], so that any edge pointing forward in time must be followed by an edge pointing backwards in time, otherwise the consecutive $(1+1)$-dimensional edges would be indistinguishable from one another. This implies that the vertices of the contour have the following simple light-cone representation:

$$
x_{2i} = (x_i^{+}, x_i^{-}) , \qquad x_{2i+1} = (x_i^{+}, x_{i+1}^{-}) , \qquad i = 1, \ldots, n .
\tag{2.3}
$$

The cross-ratios $u_{ij}^{\pm}$ appearing on the right hand side of (2.2) are functions of only either x^{+} or x^{-} light-cone coordinates, they are defined via,

$$
u_{ij}^{+} := \frac{x_{ij+1}^{+}\, x_{i+1j}^{+}}{x_{ij}^{+}\, x_{i+1j+1}^{+}} , \qquad u_{ij}^{-} := \frac{x_{ij+1}^{-}\, x_{i+1j}^{-}}{x_{ij}^{-}\, x_{i+1j+1}^{-}} ,
\tag{2.4}
$$

and as such, these cross-ratios are essentially made from one-dimensional distances. It is easy to check that this results in the following simple identity

$$
(1 - u_{ij+1}^{\pm})(1 - u_{i+1j}^{\pm}) = (1 - 1/u_{ij}^{\pm})(1 - 1/u_{i+1j+1}^{\pm})
\tag{2.5}
$$

$$
u_{i,i+1} = u_{i+1,i} = 0 \qquad u_{i,i} = \infty .
\tag{2.6}
$$

This identity can be verified directly. It can also be understood geometrically as follows from Figure 2.

Also note that this equation is precisely the AdS_3 Y-system equation of [21], where the Y's of [21] (evaluated at $\zeta = 0$) are associated with the cross-ratios as

$$
u_{k,-k-1}^{+} = \frac{Y_{2k}}{1 + Y_{2k}} \qquad u_{k,-k-2}^{-} = \frac{Y_{2k+1}}{1 + Y_{2k+1}} .
\tag{2.7}
$$

We will thus refer to (2.5) as the Y-system from now on.

3 Wilson loops for two-dimensional kinematics depend on simple logs only

Here we start assembling evidence that there is an enormous simplification for Wilson loops in $D = 4 - 2\epsilon$ dimensions when their polygonal contours are restricted to two
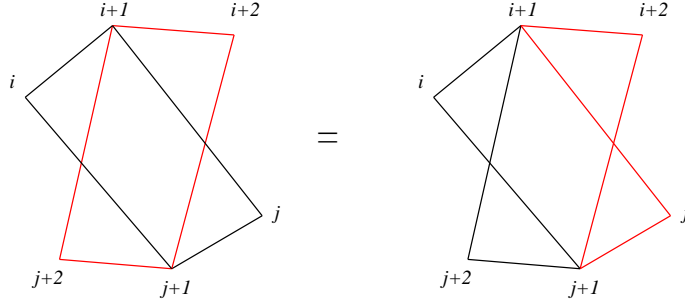
Figure 2: Figure illustrating equation (2.5). On the left we represent in black the rectangular cross-ratio $(1 - 1/u_{ij}^{\pm})$ and in red the rectangular cross-ratio $(1 - 1/u_{i+1\,j+1}^{\pm})$. On the right, on the other hand, in black we show the "crossed" cross-ratio $1 - u_{ij+1}$ and in red $1 - u_{i+1j}^{\pm}$. Clearly the product on the left-hand side equals that on the right-hand side.

space-time dimensions. The conformal cross-ratios in the two-dimensional kinematics were defined in the previous sections. We will see below that Wilson loops in this kinematics will contain only logarithmic functions (at least at one and two loops). In particular, all the dependence on polylogarithms and other more complicated functions disappears.

The first piece of evidence occurs at one loop. The fact that the BDS expression can be simplified to depend only on logarithms in AdS_3 kinematics has previously been observed by Alday and Maldacena in [18]. We shall re-derive this result, exhibiting on the way a few useful identities which we have discovered.

The general all n expression for the one-loop result for the Wilson loop/ MHV amplitude [3, 20] is well-known. It is given as follows for $n > 4$,

$$\mathcal{M}_n^{(1)}(\epsilon) = -\frac{1}{2\epsilon^2} \sum_{i=1}^{n} \left(-\frac{t_i^{[2]}}{\mu^2} \right)^{-\epsilon} + F_n^{(1)}(\epsilon) , \qquad (3.1)$$

$$F_n^{(1)}(0) = \frac{1}{2} \sum_{i=1}^{n} g_{n,i} , \qquad (3.2)$$

where

$$g_{n,i} = -\sum_{r=2}^{[n/2]-1} \ln \left(\frac{-t_i^{[r]}}{-t_i^{[r+1]}} \right) \ln \left(\frac{-t_{i+1}^{[r]}}{-t_i^{[r+1]}} \right) + D_{n,i} + L_{n,i} + \frac{3}{2}\zeta_2 , \qquad (3.3)$$

and $t_i^{[r]} := (p_i + \cdots + p_{i+r-1})^2$ are the kinematical invariants. The functions $D_{n,i}$ and

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$L_{n,i}$ for the case at hand with even number of edges, $n = 2m$, are given by,

$$D_{2m,i} = -\sum_{r=2}^{m-2} \mathrm{Li}\left(1 - \frac{t_i^{[r]} t_{i-1}^{[r+2]}}{t_i^{[r+1]} t_{i-1}^{[r+1]}}\right) - \frac{1}{2}\mathrm{Li}\left(1 - \frac{t_i^{[m-1]} t_{i-1}^{[m+1]}}{t_i^{[m]} t_{i-1}^{[m]}}\right), \qquad (3.4)$$

$$L_{2m,i} = -\frac{1}{4}\ln\left(\frac{-t_i^{[m]}}{-t_{i+m+1}^{[m]}}\right)\ln\left(\frac{-t_{i+1}^{[m]}}{-t_{i+m}^{[m]}}\right).$$

As is well-known, the finite part of the one-loop result is of transcendental weight two and it contains both $\log^2$ terms and dilogarithms. The total contribution of all dilogs in the one-loop expression is,

$$\sum_{\{u\}} \mathrm{Li}_2\left(1 - u\right) \qquad (3.5)$$

where the sum is over all cross-ratios u_{ij}.

At eight points in two-dimensional kinematics the cross-ratios always appear in pairs u and $1 - u$. Indeed at eight-points the Y-system equation (2.5) becomes simply

$$\text{Octagon}: \qquad (1 - 1/u_{ij}^{\pm})(1 - 1/u_{i+1\,j+1}^{\pm}) = 1 \qquad (3.6)$$

giving $u_{i+1\,j+1}^{\pm} = 1 - u_{ij}^{\pm}$. This can also be seen from the explicit form for the octagon cross-ratios in the standard parametrisation [12, 18] in terms of χ^+ and χ^- variables,

$$\text{Octagon}: \qquad u_{15} = \frac{\chi^+}{1 + \chi^+} := u_{24}^+, \qquad u_{26} = \frac{\chi^-}{1 + \chi^-} := u_{13}^-, \qquad (3.7)$$

$$u_{37} = \frac{1}{1 + \chi^+} := u_{13}^+, \qquad u_{48} = \frac{1}{1 + \chi^-} := u_{24}^-,$$

$$u_{i,i+3} = 1, \qquad i = 1, \ldots, 8,$$

which immediately gives $u_{15} = 1 - u_{37}$ and $u_{48} = 1 - u_{26}$.

Then a dilog identity of Euler

$$\mathrm{Li}_2(u) = -\mathrm{Li}_2(1 - u) - \log(1 - u)\log(u) + \frac{\pi^2}{6} \qquad (3.8)$$

allows us to rewrite the one loop Wilson loop purely in terms of logarithms [18].

Remarkably, similar cancellations of dilogs occur at higher points too, albeit with the use of much more complicated identities.

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We have found numerically, with the help of the PSLQ algorithm [22], that the following general identity holds for any even n:

$$\sum_{i=1}^{n/2} \sum_{j=i+2}^{n/2-\delta_{i,1}} \text{Li}_2(1-u_{i,j}^{\pm}) + \sum_{i=1}^{n/2} \sum_{j=i+1}^{n/2} \sum_{k=j+1}^{n/2} \sum_{l=k+1}^{n/2} \log u_{i,k}^{\pm} \log u_{j,l}^{\pm} - \pi^2 \frac{(n/2-6)}{12} = 0 \tag{3.9}$$

for any $u_{ij}^{\pm} > 0$, with indices defined mod $n/2$ and satisfying the Y-system equation (2.5).

This identity then kills all logarithms in the one loop n-point Wilson loop/MHV amplitude over a two-dimensional contour. (Note that the first term in (3.9) is simply an explicit writing of the sum over all (plus or minus) cross-ratios and is equivalent to (3.4)).

To illustrate we take the next simplest example, $n = 10$. Here there are 25 cross-ratios.

$$\text{Decagon}: \qquad u_{i,i+3} = 1, \qquad i = 1, \ldots, 10, \tag{3.10}$$
$$u_{i,i+4}, \qquad\qquad i = 1, \ldots, 10,$$
$$u_{i,i+5} = 1, \qquad i = 1, \ldots, 5.$$

The $u_{i,i+4}$ group of 10 can be divided into two groups of 5 cross-ratios $u_{i\,i+2}^{\pm}$ as in (2.2) which we here define simply as $u_i^{\pm}$ to simplify the notation slightly. Each of these groups separately satisfies the constraint,

$$1 - u_i^{\pm} = (1 - 1/u_{i+2}^{\pm})(1 - 1/u_{i+3}^{\pm}) \qquad i = 1 \ldots 5, \tag{3.11}$$

which arises directly from the Y-system (2.5).

In this case the identity (3.9) becomes

$$\sum_{i=1}^{5} \left(\text{Li}_2(1-u_i) + \log(u_i) \log(u_{i+1}) \right) - \frac{\pi^2}{3} = 0 \tag{3.12}$$

for any u_i which satisfy (3.11). The indices in the above two equations are all understood to be Mod 5.

Note that the fact that the one-loop contribution to the Wilson loop depends only on logarithms in AdS_3 kinematics has previously been observed in [18] where they find that the BDS expression in these special kinematics simply reduces to (e.g. see Eq. (E.2) of Ref. [18])

$$-\frac{1}{4} \sum_{i=1}^{n/2} \sum_{j=1, j \neq i, i-1}^{n} \log \frac{x_{ji}^+}{x_{j+1,i}^+} \log \frac{x_{j,i-1}^-}{x_{j,i}^-}. \tag{3.13}$$

4 Two-loop results

At two loops the remainder function for $(1+1)$-dimensional octagons has recently been shown to be independent of polylogarithms taking a strikingly simple form [19]

$$R_8^{\text{DDS}} = -\frac{1}{2} \log(1 + \chi^+) \log(1 + \frac{1}{\chi^+}) \log(1 + \chi^-) \log(1 + \frac{1}{\chi^-}) - \frac{\pi^4}{18} \,. \qquad (4.1)$$

To prepare for the higher-n analysis we recast this expression in terms of logarithms of cross-ratios, $u_i := u_{ii+4}$,

$$R_8^{\text{DDS}} = -\frac{1}{2} \log(u_1) \log(u_2) \log(u_3) \log(u_4) - \frac{\pi^4}{18} \,. \qquad (4.2)$$

Knowing (from the previous section and from [18]) that at one-loop dilogs do cancel for all n-points, and that there are no polylogarithms at two-loops at $n = 8$ points it is natural to ask whether this simplicity extends to higher points at two loops.

At $n = 8$ points the form of the octagon remainder (4.2) is essentially fixed by multi-collinear limits and (cyclic/parity) symmetry up to a single factor, once one makes an educated guess that the final result only depends on logs of cross-ratios.

We found that conditions imposed by multi-collinear limits together with this logs-only assumption are even stronger for higher points, completely fixing the form of higher-point Wilson loops. We will now demonstrate this explicitly for the 10-point and 12-point Wilson loops .

4.1 Decagon

Define $u_i := u_{ii+4}$.

Our result for the decagon Wilson loop remainder function at 2-loops and in the $(1+1)$-dimensional kinematics is

$$R_{10} = -\frac{1}{2} \Big(\log(u_1) \log(u_2) \log(u_3) \log(u_4) + \text{cyclic} \Big) - \frac{\pi^4}{12} \,, \qquad (4.3)$$

where 'cyclic' means that the first term with labels $1, 2, 3, 4$ is accompanied by $2, 3, 4, 5$ plus eight more terms up to $10, 1, 2, 3$.

The expression above is fixed by the triple collinear limit which gives very strong constraints.[3] For concreteness take momenta p_8, p_9 and p_{10} to be collinear. Under this triple collinear limit $\mathcal{R}_{10} \to \mathcal{R}_8 + \mathcal{R}_6$ and we have that $u_5 \to 1$, $u_7 \to 0$, $u_9 \to 1$. Moreover, the cross-ratios u_1, u_2, u_3 of $\mathcal{R}_{10}$ coincide with those of $\mathcal{R}_8$, and the combination $u_4 u_{10}$ maps onto u_4 of $\mathcal{R}_8$.

In this limit the remainder function should reduce as follows

$$R_{10}(u_i) \to R_8(u_1, u_2, u_3, u_4 u_{10}) + R_6 \tag{4.4}$$

$$= -\frac{1}{2}\Big(\log(u_1) \log(u_2) \log(u_3) \log(u_4 u_{10}) \Big) - \frac{\pi^4}{12} \ . \tag{4.5}$$

This is quite a strong constraint. In particular there should be no dependence on the variables u_6, u_8 and the cross-ratios u_4 and u_{10} must appear only in the combination $u_4 u_{10}$.

The above putative remainder (4.3) has precisely these properties.

We have also verified numerically, using the technology developed in [8], that our analytic expression (4.3) is in precise agreement with the numerical results.

4.2   Dodecagon

Here there are 42 cross-ratios,

$$
\begin{array}{llll}
\text{Dodecagon}: & u_{i,i+3} = 1 \ , & i = 1, \ldots, 12 \ , & (4.6) \\
& u_{i,i+4} := u_i \ , & i = 1, \ldots, 12 \ , & \\
& u_{i,i+5} = 1 \ , & i = 1, \ldots, 12 \ , & \\
& u_{i,i+6} := v_i \ , & i = 1, \ldots, 6 \ . &
\end{array}
$$

In the triple collinear limit with collinear p_{10}, p_{11} and p_{12}, we have $u_7 \to 1$, $u_9 \to 0$, $u_{11} \to 1, v_5 \to 1$ and the remainder function should reduce as

$$R_{12}(u_i; v_i) \to R_{10}(u_1, u_2, u_3, u_4, u_5, u_6 v_6, v_1, v_2, v_3, v_4 u_{12}) + R_6 \ . \tag{4.7}$$

As previously, all dependence on u_8 and u_{10} is lost and the dependence on u_6, v_6, v_4, u_{12} appears only via $u_6 v_6$ and $v_4 u_{12}$.

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[3]Triple collinear limits in terms of remainder functions were discussed in detail in [8]. In general, as explained in [13], in the limit where $k + 1$ consecutive momenta (edges) become collinear, the remainder function transforms as $\mathcal{R}_n \to \mathcal{R}_{n-k} + \mathcal{R}_{k+4}$ where the second term on the r.h.s. arises from the corresponding splitting function.

The 12 point Wilson loop which satisfies these properties is

$$R_{12} = -1/2 \Big(\log(u_1) \log(u_2) \log(u_3) \log(u_4) + 11 \text{ cyclic}$$
$$+ \log(u_1) \log(u_2) \log(u_3) \log(v_4) + 11 \text{ cyclic}$$
$$+ \log(u_1) \log(u_2) \log(v_3) \log(v_4) + 11 \text{ cyclic}$$
$$+ \log(u_1) \log(v_2) \log(v_3) \log(v_4) + 11 \text{ cyclic}$$
$$+ \log(v_1) \log(v_2) \log(v_3) \log(v_4) + 5 \text{ cyclic} \Big) - \pi^4/9 . \qquad (4.8)$$

There is another collinear limit one can consider on R_{12}, the quintuple collinear limit. Under this limit we have

$$u_5 \to 1, \quad u_{11} \to 1, \quad v_3 \to 1, \quad v_5 \to 1, \quad v_1 \to 0 , \qquad (4.9)$$

and the remainder should split as

$$R_{12}(u; v) \to R_8(u_1, u_2, u_3, u_4 v_4 v_{12}) + R_8(u_7, u_8, u_9, u_{10} v_4 u_6) . \qquad (4.10)$$

This is a highly non-trivial check of our function R_{12} and we stress that no information from this quintuple collinear limit was used in the determination of R_{12}. Nevertheless, as one can easily check, the function R_{12} defined in (4.8) satisfies (4.10) in this collinear limit.

Finally, we have checked that the analytic expression (4.8) agrees well with a sample of numerical data points which we computed for $n = 12$.

4.3 General n-point at two loops

The above results at 8, 10 and 12 points are special cases of the following formula for general n

$$R_n = -\frac{1}{2} \Big(\sum_{\mathcal{S}} \log(u_{i_1 i_5}) \log(u_{i_2 i_6}) \log(u_{i_3 i_7}) \log(u_{i_4 i_8}) \Big) - \frac{\pi^4}{72}(n - 4) , \qquad (4.11)$$

where the sum runs over the set

$$\mathcal{S} = \Big\{ i_1, \ldots i_8 : 1 \le i_1 < i_2 < \cdots < i_8 \le n, \quad i_k - i_{k-1} = \text{odd} \Big\} . \qquad (4.12)$$

This sum has the following geometrical interpretation. Represent the cross-ratios u_{ij} as lines from edge i to j of our polygonal contour and assign a label $+$ or $-$ to the line corresponding to whether i, j are both odd or both even (corresponding to whether
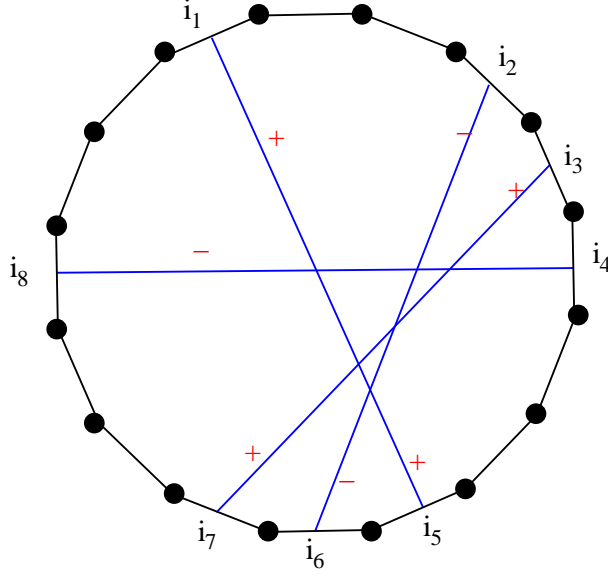
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Figure 3: The two loop n-point remainder function is given in (4.11). It consists of a sum of terms of the form $\log(u_{i_1 i_5}) \log(u_{i_2 i_6}) \log(u_{i_3 i_7}) \log(u_{i_4 i_8})$ one term of which is represented pictorially here. One sums over all possible ways of drawing four mutually crossing lines connecting edges of the polygon, such that the parity of the lines alternates.

it is a $+$ or $-$ cross-ratio in $(1+1)$-dimensions – see (2.2).) The sum is then over all ways of drawing four mutually crossing lines connecting edges of the contour, with the parity of the lines alternating (see figure 3).

Under the triple collinear limit in which edges $n-2, n-1, n$ become collinear (and in which in fact $n-1$ becomes soft) one has

$$u_{i,n-1} \to 1, \qquad u_{1,n-3} \to 0 , \tag{4.13}$$

and the remainder function should reduce as

$$R_n(u_{ij}) \to R_{n-2}(\hat{u}_{ij}) + R_6 \tag{4.14}$$

where the $n-2$-point cross-ratios $\hat{u}_{ij}$ are defined in terms of the n-point cross-ratios in the collinear limit as

$$\hat{u}_{i\,n-2} = u_{i\,n-2} u_{i\,n} \qquad \hat{u}_{ij} = u_{ij} \ i, j \neq n-2 . \tag{4.15}$$

Indeed one can check that the above formula (4.11) does indeed reduce precisely as required by (4.14).

We should also check the correct behaviour under more general collinear limits. In the case where the $2k + 1$ edges $n - 2k, \ldots, n$ become collinear we have

$$u_{1,n-2k-1} \to 0, \quad u_{i,n-2k+1} \to 1, \quad \ldots \quad u_{i,n-1} \to 1 \qquad (2 \leq i \leq n - 2k - 2), \quad (4.16)$$

and the remainder function should reduce as

$$R_n(u_{ij}) \to R_{n-2k}(\hat{u}_{ij}) + R_{2k+4}(u'_{ij}) \tag{4.17}$$

where the cross-ratios of the reduced remainders are related to the n-point cross-ratios as

$$\hat{u}_{i,n-2k} = u_{i,n-2k} \ldots u_{i,n}, \quad \hat{u}_{ij} = u_{ij} \qquad 1 \leq i, j < n - 2k \tag{4.18}$$

$$u'_{i,2} = u_{i,2} \ldots u_{i,n-2k-2}, \quad u'_{ij} = u_{ij} \qquad 0 \geq i, j \geq -2k . \tag{4.19}$$

We have checked explicitly (using a computer) in numerous non-trivial examples that our remainder function (4.11) satisfies all the correct multi-collinear limits. In the appendix we explain with the help of some pictures why the multi-collinear limit works.

4.4   Comparison with the n-point regular polygons

Using the general formula for the n-point remainder we can now specialise to the Z_n-symmetric or regular polygons studied previously at strong coupling [18] and numerically at weak coupling [12]. The cross-ratios for the regular polygons take the form as in [12]

$$\begin{aligned} u_{ij} &= 1 , & i - j &= \text{odd} , \\ u_{ij} &= 1 - \left(\frac{\sin \frac{2\pi}{n}}{\sin \frac{\pi(i-j)}{n}} \right)^2 , & i - j &= \text{even} . \end{aligned} \tag{4.20}$$

Since these regular polygons can be embedded into $(1 + 1)$-dimensions (up to a conformal transformation), the cross-ratios in (4.20) are of the form required by (2.2) and explicitly solve the Y-system equation (2.5).

Plugging these into the general remainder function (4.11) gives us the n-point remainder function for the regular polygons. Table 1 displays these results against the corresponding numerical results computed in [12]. One can see that they perfectly agree to 3 digits. This agreement provides additional evidence in favour of our main result (4.11).

The regular polygons considered above live in $(2+1)$-dimensions and are obtained from special $(1 + 1)$-dimensional polygons with a conformal transformation. They

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n	8	10	12	14	16	18	20	22	24	30
$\mathcal{R}_n^{\mathrm{num}}$	-5.528	-8.386	-11.262	-14.145	-17.035	-19.926	-22.821	-25.717	-28.614	-37.311
$\mathcal{R}_n^{\mathrm{an}}$	-5.52703	-8.38554	-11.2606	-14.1444	-17.0334	-19.9257	-22.8204	-25.7167	-28.6142	-37.3114

Table 1: *Numerical values for the regular n-point Wilson loop 2-loop remainder function against the corresponding analytic values obtained from our general analytic formula (4.11).*

are regular in the sense that they possess the discrete Z_n symmetry in 2 spatial dimensions. One can also consider other less restricted examples of polygons with discrete symmetries, such as $Z_{n/4}$-symmetric polygons (for n divisible by 4). All such cases provide convenient settings for numerical verification of the general analytic expression (4.11).

5   Conclusions

Based on a combination of analytic and numerical techniques as well as on the use of multi-collinear limits, we derived a compact analytic formula for Wilson loops with contours embedded in $(1+1)$-dimensional subspace of Minkowski space. Our two-loop expression for the remainder function (4.11) holds for any number of external momenta (or edges), thus generalising to arbitrary n the recent result for 2-dimensional octagons computed in [19].

In our view, it is a striking feature of the all-n results presented here that while the direct analytic NLO computation (when available) *at the first instance* gives a very complicated answer, the end result (4.11) is a compact logs-only expression. One expects that there must be an alternative simpler way to compute weakly coupled Wilson loops or indeed the scattering amplitudes themselves. It is known that at strong coupling there is an integrable theory set-up at work [18, 21, 23–25]. It would be interesting to find an appropriate alternative formalism applicable at weak coupling as well.

At the same time, the simplicity of one-loop and two-loop results strongly suggests that similarly simple formulae (involving only logarithms) should apply to three-loops and beyond in special kinematics. In particular a simple generalisation of (4.11) involving six logarithms is immediately apparent. Diagrammatically one simply writes down six mutually crossing lines, alternately even and odd (ie joining even and odd sides) in all possible ways. This automatically satisfies all possible collinear limits (when an appropriate n dependent constant is added) as can be seen from similar arguments to those in the appendix and has been checked extensively using a computer.

Indeed an L-loop generalisation is now apparent also, simply involving $4L$ logarithms. However, such a three loop expression will vanish at 8 and 10 points (and the L loop generalisation will only be non-zero from $4L$ points) as there are not enough edges to write it down. If these were the only possible L-loop structures, then this would mean that the all orders octagon remainder in $1 + 1$ dimensions would be equal to the two loop expression (up to multiplicative and additive, coupling dependent constants.) We now know that this is not the case (and so there must be additional structures occurring at higher loops on top of these) since the analytic expression at strong coupling [18] does not have the same form as at weak coupling [19]. Intriguingly, however the strong coupling result does agree numerically to quite high accuracy with the two loop one [12].

Acknowledgements

Appendix: multi-collinear limits

In this appendix we explain in some detail why our n-point 2 loop expression correctly reproduces the multi-collinear limits. We illustrate this in the case of the quintuple collinear limit acting on the 12 point Wilson loop, but the discussion generalises in a straightforward way to an arbitrary multi-collinear limit acting on an arbitrary polygon in a straightforward manner. We take edges $8, 9, 10, 11, 12$ (which we collectively refer to as 'A' to approach the collinear limit, with edges 9 and 11 also becoming soft. In this limit the expected collinear behaviour is $R_{12} \to R_8 + R_8$ [8, 13]
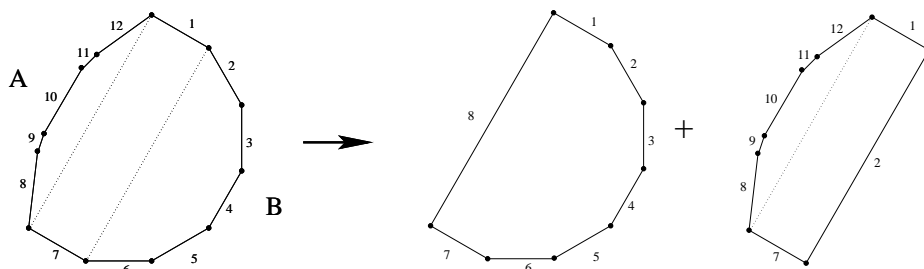
In this diagram the edges $8, 9, 10, 11, 12$ are becoming collinear, approaching the dashed edge. Note that there is a complete symmetry between this case and that of taking the edges $2, 3, 4, 5, 6$ (which we will call B) collinear. Indeed as shown recently in [25], a conformal transformation can take us from one collinear limit to the other.

How does the n-point result (4.11) respect this collinear limit? Let us consider different terms in the expression. As in figure 3 we will represent $\log(u_{ij})$ by a line drawn between edge i and edge j. We will call such a line 'odd' if i, j are odd and 'even' if they are even (corresponding to u_{ij} being a + or a - cross-ratio in (1+1) dimensions, see (2.2)). The two loop Wilson loop expression (4.11) then corresponds to a sum over terms with four lines.

From (4.16) we see that any term containin a single odd line between A and B vanishes in the limit, for example
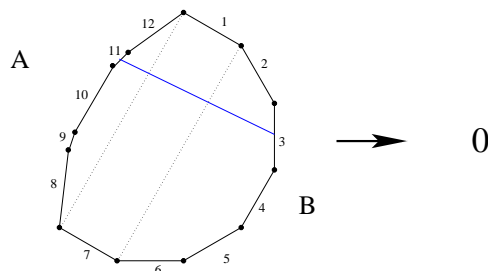
What can we say about 'even' lines between A and B? Crucially, in any given $\log^4$ term in the n-point expression (4.11) we will never have more than one such 'even' edge (between A and B) surviving in the collinear limit. This is because any given term which contains a product of two 'even' lines between edges A and B must inevitably also contain an odd line stretching between edges A and B, sandwiched between the two 'even' lines. The odd line will then vanish in the collinear limit and hence kill such a term in the expression. So 'even' lines between A and B occur alone in the collinear limit: the accompanying three lines must stretch either from A^c to A^c (the complement of A) or stretch from B^c to B^c. Indeed the three accompanying lines must either *all* be in A^c or all in B^c since if one line was in A^c and another in B^c then the lines would not intersect each other.

On the other hand 'even' lines between A and B sum up to give the respective line of the reduced Wilson loop (recall that the relevant reduced cross-ratio is a product of the original cross-ratios (4.18).) So any four lines involving an 'even' line between A and B must come accompanied by other similar terms to reproduce four lines in the reduced expression. For example,
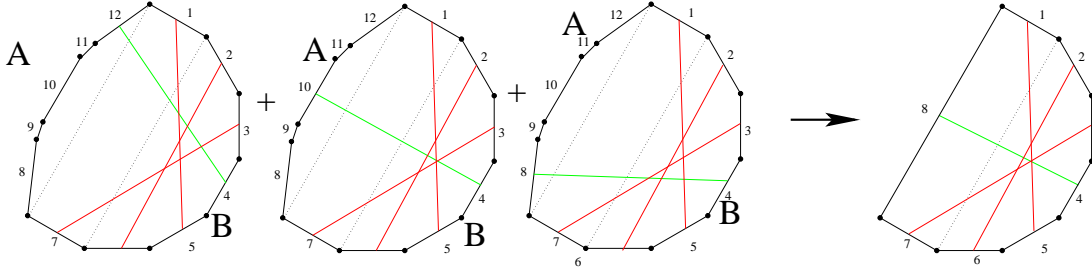
We have thus covered all situations involving lines between A and B. All other terms must consist of four lines entirely in A^c or entirely in B^c (as before we can not have a mixture of such lines as they would not intersect.) These are not present in our example but can occur at higher points. These terms simply reproduce corresponding terms in the reduced Wilson loops. Finally there is only one more case to consider, the single line between the two edges in $A^c \cap B^c$ (ie between edges 1 and 7 in our example.) This term should not occur in the collinear limit, and indeed this is the case. The reason is that such a line will inevitably have to be accompanied by three lines between A and B (since they have to cross each other.) Two of these line will be 'even', but the other one will be odd and hence vanish in the collinear limit.

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References

[1] L. F. Alday and J. Maldacena, *Gluon scattering amplitudes at strong coupling,* JHEP **0706** (2007) 064, `0705.0303` [hep-th].

[2] J. M. Drummond, G. P. Korchemsky and E. Sokatchev, *Conformal properties of four-gluon planar amplitudes and Wilson loops,* Nucl. Phys. B **795** (2008) 385, `0707.0243` [hep-th].

[3] A. Brandhuber, P. Heslop and G. Travaglini, *MHV Amplitudes in N=4 Super Yang-Mills and Wilson Loops,* Nucl. Phys. B **794** (2008) 231, `0707.1153` [hep-th].

[4] J. M. Drummond, J. Henn, G. P. Korchemsky and E. Sokatchev, *On planar gluon amplitudes/Wilson loops duality,* Nucl. Phys. B **795** (2008) 52, `0709.2368` [hep-th].

[5] J. M. Drummond, J. Henn, G. P. Korchemsky and E. Sokatchev, *Conformal Ward identities for Wilson loops and a test of the duality with gluon amplitudes,* Nucl. Phys. B **826** (2010) 337 [arXiv:0712.1223 [hep-th]].

[6] Z. Bern, L. J. Dixon, D. A. Kosower, R. Roiban, M. Spradlin, C. Vergu and A. Volovich, *The Two-Loop Six-Gluon MHV Amplitude in Maximally Supersymmetric Yang-Mills Theory,* Phys. Rev. D **78**, 045007 (2008), `0803.1465` [hep-th].

[7] J. M. Drummond, J. Henn, G. P. Korchemsky and E. Sokatchev, *Hexagon Wilson loop = six-gluon MHV amplitude,* Nucl. Phys. B **815** (2009) 142, `0803.1466` [hep-th].

[8] C. Anastasiou, A. Brandhuber, P. Heslop, V. V. Khoze, B. Spence and G. Travaglini, *Two-Loop Polygon Wilson Loops in N=4 SYM,* JHEP **0905** (2009) 115, `0902.2245` [hep-th].

[9] C. Vergu, *The two-loop MHV amplitudes in N=4 supersymmetric Yang-Mills theory,* arXiv:0908.2394 [hep-th].

[10] Z. Bern, L. J. Dixon and V. A. Smirnov, *Iteration of planar amplitudes in maximally supersymmetric Yang-Mills theory at three loops and beyond,* Phys. Rev. D **72** (2005) 085001, `hep-th/0505205`.

[11] C. Anastasiou, Z. Bern, L. J. Dixon and D. A. Kosower, *Planar amplitudes in maximally supersymmetric Yang-Mills theory,* Phys. Rev. Lett. **91** (2003) 251602, `hep-th/0309040`.

[12] A. Brandhuber, P. Heslop, V. V. Khoze and G. Travaglini, *Simplicity of Polygon Wilson Loops in N=4 SYM,* JHEP **1001**, 050 (2010) [arXiv:0910.4898].

[13] P. Heslop and V. V. Khoze, *Regular Wilson loops and MHV amplitudes at weak and strong coupling,* JHEP **1006** (2010) 037 [arXiv:1003.4405 [hep-th]].

[14] V. Del Duca, C. Duhr and V. A. Smirnov, *An Analytic Result for the Two-Loop Hexagon Wilson Loop in N = 4 SYM,* arXiv:0911.5332.

[15] V. Del Duca, C. Duhr and V. A. Smirnov, *The Two-Loop Hexagon Wilson Loop in N = 4 SYM,* arXiv:1003.1702.

[16] J. H. Zhang, *On the two-loop hexagon Wilson loop remainder function in N=4 SYM,* arXiv:1004.1606 [hep-th].

[17] A. B. Goncharov, M. Spradlin, C. Vergu and A. Volovich, *Classical Polylogarithms for Amplitudes and Wilson Loops,* arXiv:1006.5703 [hep-th].

[18] L. F. Alday and J. Maldacena, *Null polygonal Wilson loops and minimal surfaces in Anti-de-Sitter space,* JHEP **0911**, 082 (2009) [arXiv:0904.0663 [hep-th]].

[19] V. Del Duca, C. Duhr and V. A. Smirnov, *A Two-Loop Octagon Wilson Loop in N = 4 SYM,* arXiv:1006.4127 [hep-th].

[20] Z. Bern, L. J. Dixon, D. C. Dunbar and D. A. Kosower, *One Loop N Point Gauge Theory Amplitudes, Unitarity And Collinear Limits,* Nucl. Phys. B **425** (1994) 217, `hep-ph/9403226`.

[21] L. F. Alday, D. Gaiotto and J. Maldacena, *Thermodynamic Bubble Ansatz,* arXiv:0911.4708.

[22] H. R. P. Ferguson, and D. H. Bailey, *A Polynomial Time, Numerically Stable Integer Relation Algorithm.* RNR Techn. Rept. **RNR-91-032** (1992).

[23] Y. Hatsuda, K. Ito, K. Sakai and Y. Satoh, *Thermodynamic Bethe Ansatz Equations for Minimal Surfaces in AdS$_3$,* arXiv:1002.2941 [hep-th].

[24] L. F. Alday, J. Maldacena, A. Sever and P. Vieira, *Y-system for Scattering Amplitudes,* arXiv:1002.2459.

[25] L. F. Alday, D. Gaiotto, J. Maldacena, A. Sever and P. Vieira, *An Operator Product Expansion for Polygonal null Wilson Loops,* arXiv:1006.2788 [hep-th].

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